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LETTER TO THE EDITOR

Symmetries of factorization chains for the discrete Schrödinger equation

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Abstract. Factorization chains for the one-dimensional discrete Schrödinger equation (or the three-term recurrence relation for orthogonal polynomials) defining discrete-time Toda and Volterra lattices are considered. Discrete symmetries, arising from a freedom in the intermediate steps of corresponding double spectral transformations, are described. Some consequences for orthogonal polynomials are discussed.

This letter is devoted to the one-dimensional discrete Schrödinger equation and the three-term recurrence relation for orthogonal polynomials. The main goal of the present and other related works is to reconsider the theory of special functions defined upon the standard differential and finite-difference Schrödinger equations as a subcase of the general theory of self-similar systems. In principle, the corresponding point of view could be applied to more complicated spectral problems, but our attention is paid to the Schrödinger equations because they play a prominent role in quantum mechanics, the theory of solitons and some other branches of physics. Within this approach the 'exactly solvable potentials' and the associated special functions appear as systems admitting additional similarity symmetries.

The factorization method [1–3] is a basic object in the following considerations. It is tied to the Darboux transformations for Sturm–Liouville problems. In the theory of orthogonal polynomials, related constructions were considered by Christoffel and Geronimus. Because under the corresponding transformations the spectrum of taken problems is changed in a simple controllable way, it is convenient to refer to all of them as spectral transformations. Not so long ago the factorization method found an interpretation within the supersymmetric quantum mechanics and its parasupersymmetric generalization [4]. The latter idea was useful in the discovery of an infinite hierarchy of self-similar potentials defining a new class of q-special functions (see [5] and references therein).

An interesting Weyl group symmetry of the factorization chain for the differential Schrödinger equation was found in [6]. The main result of this paper consists in deriving an explicit form of its analogue for the discrete Schrödinger equation (or the three-term recurrence relation for orthogonal polynomials). A discrete factorization chain and some of its symmetry reductions (q-periodic closures) were described in [7]. In [8] it was shown that this chain defines a discrete-time Toda lattice, a very simple discrete-time Volterra lattice has been derived and the Askey–Wilson polynomials (the most general set of classical

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orthogonal polynomials) were shown to define a class of solutions of these lattices. Our notation below is close to that of [5, 8].

Let us start from a brief description of the Weyl group symmetry in the standard differential case [6]. Consider an infinite chain of eigenvalue problems

$$H_j\psi_j(x) \equiv -\psi_j''(x) + u_j(x)\psi_j(x) = \lambda\psi_j(x), \qquad j \in \mathbb{Z} \tag{1}$$

such that the nearest neighbours are related to each other by the forward discrete-time evolution law

$$\psi_j(x) = \frac{L_j\psi_{j-1}(x)}{\lambda - \lambda_j} \qquad L_j \equiv -\frac{\mathrm{d}}{\mathrm{d}x} - f_j(x) \tag{2}$$

where $f_j(x)$ are some functions, called superpotentials, and λ_j are some constants. It is convenient to assume that x is a complex variable and that no particular boundary conditions are imposed upon the equations (1). The backward time evolution is defined as

$$\psi_{j-1}(x) = R_j\psi_j(x) \qquad R_j \equiv \frac{\mathrm{d}}{\mathrm{d}x} - f_j(x). \tag{3}$$

The compatibility conditions of (1)–(3) are expressed as the intertwining relations $H_{j-1}R_j = R_jH_j$ and $L_jH_{j-1} = H_jL_j$. Their resolution yields the factorization conditions $H_j = L_jR_j + \lambda_j$ and $H_{j-1} = R_jL_j + \lambda_j$, or

$$u_j(x) = f_j^2(x) + f_j'(x) + \lambda_j \qquad u_{j-1}(x) = u_j(x) - 2f_j(x). \tag{4}$$

In the general setting this gives the abstract factorization chain

$$R_jL_j + \lambda_j = L_{j-1}R_{j-1} + \lambda_{j-1} \tag{5}$$

which, in its particular differential operator realization given above, leads to the following differential-difference equation [2]:

$$f_j'(x) + f_{j-1}'(x) + f_{j-1}^2(x) - f_j^2(x) = \lambda_j - \lambda_{j-1}. \tag{6}$$

Actually the explicit form of L_j and R_j in (5) does not play a crucial role for the applicability of the factorization method—below we consider the case when these operators are finite-difference operators of the first order, but there are also cases when they have a mixed, differential-difference form. In the latter two cases only the nature of spectral problem under investigation is changed, for example the differential Schrödinger equation is replaced by its finite-difference analogue.

By investigating symmetries of the chain (6) and making the appropriate symmetry reductions one can find potentials obeying additional symmetries. These symmetries allow us to derive many properties of the solutions of corresponding Schrödinger equations characteristic to special functions. A very general class of such self-similar potentials is defined by a combination of the discrete symmetry $j \to j + N$, where N is some integer, and the affine transformations. More precisely, it is described by the reduction $f_{j+N}(x) = qf_j(qx + a), \lambda_{j+N} = q^2\lambda_j$, where q and a are some complex parameters. In this way one gets a set of special functions, which can be interpreted as q-deformations of the Painlevé-type transcendents, intrinsically related to the finite-gap potentials. From the algebraic point of view these systems are associated with the representations of some polynomial algebra, which for $N = 2$ coincides with the quantum algebra $su_q(1, 1)$. For more details we refer to the review [5].

In addition to the global discrete symmetry $j \to j + N$ mentioned above, there is a local one associated with a nice motion of polygons [6]. It appears from the fact that a sequence of K Darboux transformations with the parameters $\lambda_j, \lambda_{j+1}, \ldots, \lambda_{j+K-1}$ yields the same final result independent of the order of intermediate steps. This is obvious from the Wronskian representation of the K-step transformations [3]. As a result, there appears a non-trivial symmetry of superpotentials. In the simplest case, $K = 2$, it has the form [6]

$$\tilde{f}_{j-1} = f_{j-1} - \frac{\lambda_j - \lambda_{j-1}}{f_j + f_{j-1}} \qquad \tilde{f}_j = f_j + \frac{\lambda_j - \lambda_{j-1}}{f_j + f_{j-1}} \tag{7}$$

$$\tilde{\lambda}_{j-1} = \lambda_j \qquad \tilde{\lambda}_j = \lambda_{j-1} \tag{8}$$

all other ingredients remaining untouched:

$$\tilde{f}_k(x) = f_k(x) \qquad \tilde{\lambda}_k = \lambda_k \qquad k \neq j, j-1.$$

Indeed, the requirement

$$\frac{L_j L_{j-1}}{(\lambda - \lambda_j)(\lambda - \lambda_{j-1})} = \frac{\tilde{L}_j \tilde{L}_{j-1}}{(\lambda - \tilde{\lambda}_j)(\lambda - \tilde{\lambda}_{j-1})} \tag{9}$$

leaves two options for transformations of λ_j's—either $\tilde{\lambda}_j = \lambda_j$, $\tilde{\lambda}_{j-1} = \lambda_{j-1}$, which does not induce non-trivial transformations, or (8). Other relations (7) can be found from (9) or by the direct resolution of constraints stemming from the factorization chain.

Let us denote generators of the above symmetry as B_j, i.e. $B_j f_k(x) = \tilde{f}_k(x)$, etc. They form a Weyl group with the composition laws [6]

$$B_j^2 = 1 \qquad B_i B_j = B_j B_i \qquad i \neq j \pm 1 \qquad (B_{j-1} B_j)^3 = 1. \tag{10}$$

Let us now derive explicitly an analogue of the above symmetry for factorization chains for the discrete Schrödinger equation [7, 8]. Consider an infinite set of spectral problems:

$$H_j \psi_n^j \equiv \psi_{n+1}^j + u_n^j \psi_{n-1}^j + b_n^j \psi_n^j = \lambda \psi_n^j \qquad n, j \in \mathbb{Z}. \tag{11}$$

If one imposes the boundary conditions $\psi_{-1}^j(\lambda) = 0$, $\psi_0^j(\lambda) = 1$, then by the Favard theorem $\psi_n^j(\lambda)$, $n = 0, 1, \ldots,$ define a set of orthogonal polynomials of the variable λ. In analogy with the continuous case, define the forward time step by the Christoffel's spectral transformation

$$\psi_n^{j+1} = \frac{\psi_{n+1}^j + C_n^{j+1} \psi_n^j}{\lambda - \lambda_{j+1}} \equiv \frac{L_{j+1} \psi_n^j}{\lambda - \lambda_{j+1}} \tag{12}$$

which is known in the theory of orthogonal polynomials as a definition of *kernel* polynomials [9]. The backward discrete-time flow has the form

$$\psi_n^{j-1} = \psi_n^j + A_n^j \psi_{n-1}^j \equiv R_j \psi_n^j. \tag{13}$$

In (12) and (13) A_n^j and C_n^j are discrete superpotentials. The compatibility conditions again yield factorizations of the operators H_j, which are equivalent now to the relations

$$u_n^j = A_n^j C_n^j \qquad b_n^j = A_{n+1}^j + C_n^j + \lambda_j. \tag{14}$$

The abstract factorization chain (5) in this case is equivalent to the following coupled system of nonlinear finite-difference equations:

$$A_n^j C_{n-1}^j = A_n^{j-1} C_n^{j-1} \qquad A_n^j + C_n^j + \lambda_j = A_{n+1}^{j-1} + C_n^{j-1} + \lambda_{j-1}. \tag{15}$$

In [8] these equations were shown to define a discrete-time Toda lattice. Note that the first integrals λ_j 'measure' the breaking of isospectrality of the corresponding discrete-time flow.

In [7] some symmetries of the chain (15) were described and the following q-periodic closure has been found:

$$A_n^{j+N} = q A_{n+k}^j \qquad C_n^{j+N} = q C_{n+k}^j \qquad \lambda_{j+N} = q \lambda_j. \tag{16}$$

Here k is an arbitrary integer. If n is considered as a continuous (complex) variable, then k may take arbitrary values as well. The closure (16) is associated with certain classical and more general orthogonal polynomials whose recurrence coefficients are related to q-analogues of some discrete Painlevé-type transcendents. In this case, the discrete spectrum of H_j is formally given by a superposition of up to N geometric series.

Using the same line of reasoning as in the continuous case, for example using the relation (9), after some straightforward calculations the author has derived the corresponding symmetry of (15).

Proposition 1. The discrete factorization chain (discrete-time Toda lattice) (15) has the following discrete symmetry

$$\tilde{A}_n^{j-1} = A_n^{j-1} + \frac{(\lambda_j - \lambda_{j-1})(A_n^j + A_n^{j-1})}{C_{n-1}^j + C_n^{j-1} - A_n^j - A_n^{j-1}}$$

$$\tilde{A}_n^{j} = A_n^{j} - \frac{(\lambda_j - \lambda_{j-1})(A_n^j + A_n^{j-1})}{C_{n-1}^j + C_n^{j-1} A_n^j A_n^{j1}} \tag{17}$$

$$\tilde{C}_n^{j-1} = C_n^{j-1} - \frac{(\lambda_j - \lambda_{j-1})(C_{n-1}^j + C_n^{j-1})}{C_{n-1}^j + C_n^{j-1} - A_n^j - A_n^{j-1}}$$

$$\tilde{C}_n^{j} = C_n^{j} + \frac{(\lambda_j - \lambda_{j-1})(C_n^j + C_{n+1}^{j-1})}{C_n^j + C_{n+1}^{j-1} - A_{n+1}^j - A_{n+1}^{j-1}} \tag{18}$$

$$\tilde{\lambda}_{j-1} = \lambda_j \qquad \tilde{\lambda}_j = \lambda_{j-1} \tag{19}$$

all other A_n^k, C_n^k and λ_k remaining unchanged.

In the continuous limit $x = nh, h \to 0$ (if it exists), the difference spectral transformations turn into the differential formulae (2) and (3), and the derived symmetry turns into (7) and (8). For example, it can be seen using the following expansion in the next-next-to-the-leading-terms order

$$A_n^j \to -(1 + hf_j(x)) \qquad C_n^j \to -(1 - hf_j(x) + \tfrac{1}{2}h^2(f_j^2(x) - f_j'(x) - \mu_j))$$

$$\lambda_j \to 4 - h^2\mu_j \qquad \lambda \to 4 - h^2\mu \qquad \psi_n^j \to h^{-j}\psi_j(x)$$

where μ_j and μ denote spectral parameters for the continuous Schrödinger equation. Similar to the differential case [6], the discrete symmetries of (15) $j \to j+k$ and (17)–(19) describe Bäcklund transformations for discrete Painlevé transcendents and their q-analogues defined by the self-similar reduction (16). Evidently, the symmetry (17)–(19) is expected to satisfy the group law (10), but the author was able to verify explicitly only the simplest relations (the difficulty consists in the non-local character of transformations of A_n^j and C_n^j in the variable n).

Let us turn now to the discrete-time Volterra lattice

$$D_n^j(D_{n-1}^j - \beta_j) = D_n^{j-1}(D_{n+1}^{j-1} - \beta_{j-1}) \tag{20}$$

which has been derived in [8]. The discrete-time analogue of the relation between solutions of the Toda and Volterra chains has the following form [8]:

$$A_n^j = D_{2n}^j D_{2n+1}^j \qquad C_n^j = (D_{2n+1}^j - \beta_j)(D_{2n+2}^j - \beta_j) \tag{21}$$

$$\lambda_j = \text{constant} - \beta_j^2.$$

There is also a second similar set of formulae

$$A_n^j = D_{2n-1}^j D_{2n}^j \qquad C_n^j = (D_{2n}^j - \beta_j)(D_{2n+1}^j - \beta_j) \tag{22}$$

with the same expression for λ_j via β_j.

From the even–odd index representation of superpotentials A_n^j and C_n^j via D_n^j, one may expect a similar splitting for symmetry transformations of the chain (20). After some tedious calculations the author has proven the following proposition (no symbolic manipulation computer programs have been used in the calculations).

Proposition 2. The discrete symmetry associated with a freedom in intermediate steps of double discrete-time shifts in the discrete-time Volterra lattice (20) with $\beta_j \neq 0$ for any j has the form

$$\tilde{D}_n^j = \frac{1}{\beta_{j-1}}\left(\beta_j D_n^j + \frac{(\beta_j^2 - \beta_{j-1}^2)(D_n^j D_{n+1}^j + D_n^{j-1} D_{n+1}^{j-1})}{\beta_j(\beta_j - D_{n-1}^j - D_{n+1}^j) + \beta_{j-1}(\beta_{j-1} - D_n^{j-1} - D_{n+2}^{j-1})}\right) \tag{23}$$

$$\tilde{D}_n^{j-1} = \frac{1}{\beta_j}\left(\beta_{j-1} D_n^{j-1} - \frac{(\beta_j^2 - \beta_{j-1}^2)(D_{n-1}^j D_n^j + D_{n-1}^{j-1} D_n^{j-1})}{\beta_j(\beta_j - D_{n-2}^j - D_n^j) + \beta_{j-1}(\beta_{j-1} - D_{n-1}^{j-1} - D_{n+1}^{j-1})}\right). \tag{24}$$

The change of spectral parameters β_j is as follows:

$$\tilde{\beta}_j = \beta_{j-1} \qquad \tilde{\beta}_{j-1} = \beta_j. \tag{25}$$

All other variables $D_n^k, \beta_k, k \neq j, j-1$, remain unchanged. Actually, one can reverse the signs of $\tilde{D}^j$, $\tilde{D}_n^{j-1}$ simultaneously with the signs of $\tilde{\beta}_j, \tilde{\beta}_{j-1}$, but this symmetry is obvious from the structure of the chain (20).

The transformation laws for superpotentials A_n^j and C_n^j of the discrete-time Toda lattice follow from the maps (21) and (22).

If $\beta_{j-1} = 0$ (j is fixed), there appears a curious freedom:

$$\tilde{D}_n^j = \left(a_j \pm (-1)^n\sqrt{a_j^2 - 1}\right)\frac{D_n^{j-1}(D_n^j - \beta_j)}{D_{n-1}^j - \beta_j} \qquad \tilde{D}_n^{j-1} = D_{n-1}^j \tag{26}$$

where a_j is an arbitrary parameter. The $\beta_{j-1} \to 0$ limit in (23) and (24) corresponds to the $a_j = 1$ choice in (26). If $\beta_j = 0$, one has

$$\tilde{D}_n^j = D_{n+1}^{j-1} \qquad \tilde{D}_n^{j-1} = \left(b_j \pm (-1)^n\sqrt{b_j^2 - 1}\right)\frac{D_n^j(D_n^{j-1} - \beta_{j-1})}{D_{n+1}^{j-1} - \beta_{j-1}} \tag{27}$$

with a different free parameter b_j. The $\beta_j \to 0$ limit in (23) and (24) corresponds to $b_j = 1$.

Suppose that $\beta_j = 0$ for arbitrary j, then the chain (20) admits an integral of the form

$$D_n^j = \left(\gamma_j \pm (-1)^n\sqrt{\gamma_j^2 - 1}\right) D_{n+1}^{j-1} \tag{28}$$

where γ_j are arbitrary constants. The symmetry we are interested in now has a two-parameter freedom

$$\tilde{D}_n^j = \left(a_j \pm (-1)^n\sqrt{a_j^2 - 1}\right) D_{n+1}^{j-1} \qquad \tilde{D}_n^{j-1} = \left(b_j \pm (-1)^n\sqrt{b_j^2 - 1}\right) D_{n-1}^j.$$

The $\beta_j, \beta_{j-1} \to 0$ limits in (23) and (24) provide the transformation laws defined only upon the subspace of solutions corresponding to the $\gamma_j = \pm 1$ choices in (28).

We would like to finish this work by giving an explicit example, illustrating how the above symmetry changes orthogonal polynomials. Consider the following solution of the chain (15) [8]:

$$A_n^j = n\sinh^2\theta/2 \qquad C_n^j = (n+j-1)\cosh^2\theta/2 \qquad \lambda_j = -j+1$$

associated with a family of the Meixner polynomials $P_n^j(\lambda)$:

$$P_{n+1}^j(\lambda) + \tfrac{1}{4}n(n+j-1)\sinh^2\theta\, P_{n-1}^j(\lambda) + (n\cosh\theta + j\sinh^2\theta/2)P_n^j(\lambda) = \lambda P_n^j(\lambda) \tag{29}$$

$$P_{-1}^j(\lambda) = 0 \qquad P_0^j(\lambda) = 1.$$

In the standard (self-similar) case spectral transformations correspond to the change of the parameter $j \to j \pm 1$. Using the formulae (17) and (18) we find superpotentials, corresponding to the non-standard backward discrete-time step,

$$\tilde{A}_n^j = n\sinh^2\theta/2\left(1 + \frac{1}{n+(j-2)\cosh^2\theta/2}\right)$$

$$\tilde{C}_n^j = (n+j-1)\cosh^2\theta/2\left(1 - \frac{1}{n+1+(j-2)\cosh^2\theta/2}\right)$$

which generate the following recurrence coefficients:

$$\tilde{u}_n^{j-1} = \tilde{A}_n^j\tilde{C}_{n-1}^j = \frac{n(n+j-2)\sinh^2\theta}{4}\left(1 - \frac{1}{(n+(j-2)\cosh^2\theta/2)^2}\right) \tag{30}$$

$$\tilde{b}_n^{j-1} = \tilde{A}_n^j + \tilde{C}_n^j + \tilde{\lambda}_j = n\cosh\theta + (j-1)\sinh^2\theta/2 - \frac{(j-2)\sinh^2\theta}{4(n+(j-2)\cosh^2\theta/2)(n+1+(j-2)\cosh^2\theta/2)}. \tag{31}$$

Note that for $j > 2$ the coefficients $\tilde{u}_n^{j-1}$ are positive for $n = 0, 1, \ldots,$ which is necessary for the positivity of the measure of the orthogonal polynomials $\tilde{P}_n^{j-1}(\lambda)$. This transformation looks similar to the continuous Schrödinger equation case, when the harmonic oscillator superpotential $f_j(x) = x$, corresponding to a self-similar solution of the chain (6) with $\lambda_j = 2j$, is mapped onto the singular harmonic oscillator superpotential $\tilde{f}_j(x) = x + 1/x$ (here j is fixed due to the breakdown of self-similarity). Probably a similar interpretation in terms of the discretized harmonic oscillator problem is valid for the relation between the standard Meixner polynomials (or the more simple Charlier polynomials) and the above singular system.

One essential point has not been touched on in this work, namely, the geometric interpretation of the transformations (17)–(19) for the discrete-time Toda lattice and (23)–(25) for the discrete-time Volterra lattice similar to the one in [6]. This is a matter for separate consideration.

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